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Additivity of Jordan maps on Jordan algebras ☆

Peisheng Ji

Department of Mathematics, QingDao University, QingDao 266071, PR China

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Let A and B be two Jordan algebras. In this paper, we investigate the additivity of maps ϕ from A onto B that are bijective and satisfy

$$\phi(a \circ b) = \phi(a) \circ \phi(b),$$

for all $a, b \in A$. If A contains an idempotent which satisfies some conditions, then ϕ is additive. This result generalizes all results about additivity of Jordan maps in [1,3,4,5,6,7,8,9,11].

1. Introduction

Throughout this paper all algebras will be algebras over a field $\mathcal{F}$ of characteristic not two. Note that we do not assume algebras to be associative or commutative unless explicitly stated. Let A be an algebra with the product written $(a, b) \longrightarrow a \circ b$. A is called a Jordan algebra if the following two identities are satisfied for all $a, b \in A$:

$$a \circ b = b \circ a,$$
$$a \circ (b \circ a^2) = (a \circ b) \circ a^2.$$

Consider any associative algebra A. If $a, b \in A$, let $a \circ b = \frac{1}{2}(ab + ba)$. Then A^J, which by definition is the vector space A with the product $\circ$, is a Jordan algebra. By a Jordan subalgebra of A we mean a

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E-email address: jipeish@yahoo.com

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subalgebra of A^J, i.e. a linear subspace of A which is closed under the product $\circ$. Any Jordan algebra isomorphic to a Jordan subalgebra of A will be called a special Jordan algebra.

In the following, denote by $\circ$ the Jordan product on Jordan algebra. For a general discussion of Jordan algebras we refer to [2].

Definition 1. Let A and B be Jordan algebras. Consider a bijection $\phi : A \longrightarrow B$. If

$$\phi(x \circ y) = \phi(x) \circ \phi(y) \quad \text{for } x \text{ and } y \text{ of } A,$$

then we call ϕ a Jordan map.

It is an interesting problem to study the interrelation between the multiplicative and the additive structure of a ring or algebra. It is Martindale who first established a condition on a ring R such that every multiplicative bijective map on R is additive [10, Theorem]. Recently, the question of whether (general) Jordan maps on associative algebras or rings are additive is studied by many mathematicians (cf. [3-5,7-9,11]). In [1], An and Hou proved that every Jordan map on $B(H)_s$ is additive using an approach in term of spectral theory. In [6], we proved that every Jordan map from a standard Jordan operator algebra onto an arbitrary Jordan algebra is additive using a purely algebraic approach, which generalizes the results in [1]. Note that all algebras in those papers are special Jordan algebras. It is an interesting problem to study the additivity of Jordan maps between general Jordan algebras. In this paper, we prove that every Jordan map $\phi : A \longrightarrow B$ from Jordan algebra A onto Jordan algebra B is additive if A contains an idempotent which satisfies some conditions. This result generalizes almost all results about additivity of Jordan maps in [1,3-9,11].

2. Preliminaries

Some notations and facts in this section can be found in [2]. Throughout this section A will be a Jordan algebra. For $a \in A$, we define the multiplication operator $T_a : A \longrightarrow A$ by

$$T_a b = a \circ b,$$

for $b \in A$. Two elements a, b in A are said to operator commute if the operators T_a, T_b commute, i.e. if $(a \circ c) \circ b = a \circ (c \circ b)$ for all $c \in A$.

For $a, b, c \in A$, we define the Jordan triple product $\{abc\}$ of a, b, c by

$$\{abc\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b.$$

We shall write $U_{a,c}$ for the operator

$$U_{a,c}(b) = \{abc\},$$

and $U_a = U_{a,a}$. It is well known that $U_a = 2T_a^2 - T_{a^2}$.

Let p be an idempotent in A, i.e. $p^2 = p$. Clearly, if $a \in A$ operators commute with p, then $U_p T_a = T_a U_p$. Let

$$U_0 = \tau + U_p - 2T_p \quad \text{and} \quad U_{\frac{1}{2}} = 2(T_p - U_p),$$

where $\tau : A \longrightarrow A$ is the identity map of A. Clearly, if A has identity element 1, then $U_0 = U_{1-p}$ and $U_{\frac{1}{2}} = 2U_{p,1-p}$.

Theorem 2.1 (Macdonald's theorem [2]). *Any polynomial identity in three variables, with degree at most* 1 *in the third variable, and which holds in all special Jordan algebras, holds in all Jordan algebras.*

In the case that A is unital, the following results can be found in [2]. In the case that A does not have an identity element, some proof of these results is similar to the case the A is unital. Here we only give the proof that is different from the unital case.

Lemma 2.2. *Let p be a non-trivial idempotent in A. Then*

(a) $T_p = U_p + \frac{1}{2}U_{\frac{1}{2}} + 0U_0$;

(b) $U_p^2 = U_p, U_{\frac{1}{2}}^2 = U_{\frac{1}{2}}, U_0^2 = U_0$;

(c) $T_pU_p = U_pT_p = U_p, T_pU_0 = U_0T_p = 0, T_pU_{\frac{1}{2}} = U_{\frac{1}{2}}T_p = \frac{1}{2}U_{\frac{1}{2}}$;

(d) $U_pU_0 = U_0U_p = 0, U_pU_{\frac{1}{2}} = U_{\frac{1}{2}}U_p = 0, U_0U_{\frac{1}{2}} = U_{\frac{1}{2}}U_0 = 0.$

Proof. From Macdonald's theorem we have $U_p^2 = U_p$ and $T_pU_p = U_pT_p = U_p$. Now we show the other identities hold

$$\begin{aligned}
U_0^2 &= (\tau + U_p - 2T_p)(\tau + U_p - 2T_p) \\
&= \tau + U_p - 2T_p + U_p + U_p^2 - 2U_pT_p - 2T_p - 2T_pU_p + 4T_p^2 \\
&= \tau - U_p - 4T_p + 4T_p^2 = \tau - U_p - 2T_p + 2\left(2T_p^2 - T_p\right) \\
&= \tau - U_p - 2T_p + 2U_p = \tau + U_p - 2T_p = U_0, \\
U_{\frac{1}{2}}^2 &= 4(T_p - U_p)(T_p - U_p) = 4\left(T_p^2 - T_pU_p - U_pT_p + U_p^2\right) \\
&= 4\left(T_p^2 - U_p - U_p + U_p\right) = 2\left(2T_p^2 - T_p\right) + 2T_p - 4U_p \\
&= 2U_p + 2T_p - 4U_p = 2(T_p - U_p) = U_{\frac{1}{2}}, \\
T_pU_0 &= T_p(\tau + U_p - 2T_p) = T_p + T_pU_p - 2T_p^2 \\
&= U_p - \left(2T_p^2 - T_p\right) = U_p - U_p = 0.
\end{aligned}$$

Similarly, $U_0T_p = 0$.

$$\begin{aligned}
T_pU_{\frac{1}{2}} &= 2T_p(T_p - U_p) = 2T_p^2 - 2T_pU_p = 2T_p^2 - T_p + T_p - 2U_p \\
&= U_p + T_p - 2U_p = T_p - U_p = \frac{1}{2}U_{\frac{1}{2}}.
\end{aligned}$$

Similarly, $U_{\frac{1}{2}}T_p = \frac{1}{2}U_{\frac{1}{2}}$.

$$\begin{aligned}
U_pU_0 &= U_p(\tau + U_p - 2T_p) = U_p + U_p^2 - 2U_pT_p = U_p + U_p - 2U_p = 0, \\
U_pU_{\frac{1}{2}} &= 2U_p(T_p - U_p) = 2U_pT_p - 2U_p^2 = 2U_p - 2U_p = 0, \\
U_{\frac{1}{2}}U_0 &= 2(T_p - U_p)(\tau + U_p - 2T_p) = 2\left(T_p + T_pU_p - 2T_p^2 - U_p - U_p^2 - 2U_pT_p\right) \\
&= 2\left(T_p + U_p - 2T_p^2\right) = 2(U_p - U_p) = 0.
\end{aligned}$$

Similarly, we can prove that $U_0U_p = 0, U_{\frac{1}{2}}U_p = 0$ and $U_0U_{\frac{1}{2}} = 0$. □

Lemma 2.3. *Let A be a Jordan algebra and p an idempotent in A. For any $a \in A$ the following conditions are equivalent*:

(i) *a and p operator commute*,
(ii) $T_pa = U_pa$,
(iii) $a = (U_p + U_0)a$,
(iv) *a and p generate an associative subalgebra of A*.

Moreover, U_pA and U_0A are subalgebras of A, and $a \circ b = 0$ if $a \in U_pA, b \in U_0A$.

Proof. (ii)⇒(iii): From (ii), we get $(U_p + U_0)a = (U_p + \tau + U_p - 2T_p)a = a$.
(iii)⇒(i): As the proof of Lemma 2.5.5 of [2], for $b \in A$, we have

$$2[T_p, T_{p\circ b}] = [T_p, T_b], \tag{2.1}$$

where $[T_e, T_f] = T_eT_f - T_fT_e$ for all $e, f \in A$. By Lemma 2.2, we see that $p \circ (U_0a) = (T_pU_0)a = 0$ and $p \circ (U_pa) = (T_pU_p)a = U_pa$. So (2.1) implies $[T_p, T_{U_0a}] = 2[T_p, T_{p\circ(U_0a)}] = 0$ and $[T_p, T_{U_pa}] = 2[T_p, T_{p\circ(U_pa)}] = 2[T_p, T_{U_pa}]$. Thus $[T_p, T_{U_pa}] = 0$. Hence $[T_p, T_a] = [T_p, T_{(U_p+U_0)a}] = [T_p, T_{U_pa}] + [T_p, T_{U_0a}] = 0$, i.e. a operator commutes with p.

Now we prove the final statement of the lemma. Let $c, d \in U_pA$. Since U_p is an idempotent and $U_pU_0 = 0$, we have $c = (U_p + U_0)c$ and $d = (U_p + U_0)d$. Hence c and d operator commute with p, and then $U_pT_c = T_cU_p$ and$U_pT_d = T_cU_d$. Therefore $U_p(c \circ d) = U_p(T_cd) = T_c(U_pd) = c \circ d$. This shows that U_pA is closed under Jordan product and is therefore a subalgebra of A. Similarly, U_0A is also a subalgebra of A.

The rest of the proof of this lemma is similar to that of [2, Lemma 2.5.5], we leave it to reader. □

From Lemma 2.2 we conclude that U_p, $U_{\frac{1}{2}}$ and U_0 are mutually orthogonal idempotent mappings with sum τ, and that T_p has eigenvalues 1, $\frac{1}{2}$ and 0. Thus we have the following vector space decomposition:

$$A = A_1 \oplus A_{\frac{1}{2}} \oplus A_0,$$

where A_i is the eigenspace corresponding to the eigenvalue i ($i = 1, \frac{1}{2}, 0$). This is called the Peirce decomposition of A with respect to p. From Lemma 2.2, it follows that U_p, $U_{\frac{1}{2}}$ and U_0 are the projections on the direct summands $A_1, A_{\frac{1}{2}}$ and A_0 respectively.

From now, a_i means an element of A_i $\left(i = 1, \frac{1}{2}, 0\right)$. Clearly $T_pa_i = p \circ a_i = ia_i$ $\left(i = 1, \frac{1}{2}, 0\right)$.

Using Lemma 2.3, as the proof of Lemma 2.6.3, we can prove the following lemma.

Lemma 2.4. *Let A be a Jordan algebra and p a non-trivial idempotent in A. Let $A = A_1 \oplus A_{\frac{1}{2}} \oplus A_0$ be the Peirce decomposition of A with respect to p. Then we have the following multiplication rules*:

$$A_0 \circ A_0 \subseteq A_0; \quad A_1 \circ A_1 \subseteq A_1; \quad A_1 \circ A_0 = 0;$$
$$(A_1 \oplus A_0) \circ A_{\frac{1}{2}} \subseteq A_{\frac{1}{2}}; \quad A_{\frac{1}{2}} \circ A_{\frac{1}{2}} \subseteq A_1 \oplus A_0.$$

If $a \in A_1, b \in A_0$, then a and b operator commute.

3. Additivity

Our result in this section is the following.

Theorem 3.1. *Let A and B be Jordan algebras and p a non-trivial idempotent in A. Let $A = A_1 \oplus A_{\frac{1}{2}} \oplus A_0$ be the Peirce decomposition of A with respect to p. If A satisfies the following conditions*:

(i) *Let $a_i \in A_i (i = 1, 0)$. If $a_i \circ t_{\frac{1}{2}} = 0$ for all $t_{\frac{1}{2}} \in A_{\frac{1}{2}}$, then $a_i = 0$;*
(ii) *Let $a_0 \in A_0$. If $a_0 \circ t_0 = 0$ for all $t_0 \in A_0$, then $a_0 = 0$;*
(iii) *Let $a_{\frac{1}{2}} \in A_{\frac{1}{2}}$. If $a_{\frac{1}{2}} \circ t_0 = 0$ for all $t_0 \in A_0$, then $a_{\frac{1}{2}} = 0$;*

then every Jordan map from A onto B is additive.

The main technique we will use is the following argument which will be termed a "standard argument". It should be mentioned that this method is modeled by Lu in [8]. Suppose, $x, y, s \in A$ are such that $\phi(s) = \phi(x) + \phi(y)$. Multiplying this equality by $\phi(t)$, we get

$$\phi(s) \circ \phi(t) = \phi(x) \circ \phi(t) + \phi(y) \circ \phi(t).$$

It follows that

$$\phi(s \circ t) = \phi(x \circ t) + \phi(y \circ t).$$

Moreover, if

$$\phi(x \circ t) + \phi(y \circ t) = \phi(x \circ t + y \circ t),$$

then by injectivity of ϕ, we have that

$$s \circ t = (x + y) \circ t.$$

The proof will be organized in a series of lemmas. We begin with the following trivial one.

Lemma 3.2. $\phi(0) = 0$.

Proof. Since ϕ is surjective, there exists an $x \in A$ such that $\phi(x) = 0$. Therefore $\phi(0) = \phi(x \circ 0) = \phi(x) \circ \phi(0) = 0 \circ \phi(0) = 0$. $\square$

Lemma 3.3. *Let $a_i \in A_i, i = 1, \frac{1}{2}, 0$. Then $\phi(a_1 + a_{\frac{1}{2}} + a_0) = \phi(a_1) + \phi(a_{\frac{1}{2}}) + \phi(a_0)$.*

Proof. Since ϕ is surjective, we can find an element $s = s_1 + s_{\frac{1}{2}} + s_0 \in A$ such that

$$\phi(s) = \phi(a_1) + \phi(a_{\frac{1}{2}}) + \phi(a_0). \tag{3.1}$$

For p, applying a standard argument to (3.1), we get

$$\phi(p \circ s) = \phi(p \circ a_1) + \phi(p \circ a_{\frac{1}{2}}) + \phi(p \circ a_0).$$

Since $p \circ s = T_p s = s_1 + \frac{1}{2} s_{\frac{1}{2}}$, $p \circ a_1 = T_p a_1 = a_1$, $p \circ a_{\frac{1}{2}} = T_p a_{\frac{1}{2}} = \frac{1}{2} a_{\frac{1}{2}}$ and $p \circ a_0 = 0$, the equation above implies

$$\phi\left(s_1 + \frac{1}{2} s_{\frac{1}{2}}\right) = \phi(a_1) + \phi\left(\frac{1}{2} a_{\frac{1}{2}}\right) + \phi(0) = \phi(a_1) + \phi\left(\frac{1}{2} a_{\frac{1}{2}}\right). \tag{3.2}$$

For $t_0 \in A_0$, applying a standard argument to (3.2) and using Lemma 2.3, we have

$$\phi\left(\frac{1}{2} t_0 \circ s_{\frac{1}{2}}\right) = \phi\left(t_0 \circ \left(s_1 + \frac{1}{2} s_{\frac{1}{2}}\right)\right) = \phi(t_0 \circ a_1) + \phi\left(\frac{1}{2} t_0 \circ a_{\frac{1}{2}}\right) = \phi\left(\frac{1}{2} t_0 \circ a_{\frac{1}{2}}\right).$$

Therefore, $\frac{1}{2} t_0 \circ s_{\frac{1}{2}} = \frac{1}{2} t_0 \circ a_{\frac{1}{2}}$ for every $t_0 \in A_0$. It follows from Theorem 3.1(iii) that $s_{\frac{1}{2}} = a_{\frac{1}{2}}$.

For $t_{\frac{1}{2}}$, applying a standard argument to (3.2), we have

$$\phi\left(t_{\frac{1}{2}} \circ s_1 + \frac{1}{2} t_{\frac{1}{2}} \circ s_{\frac{1}{2}}\right) = \phi(t_{\frac{1}{2}} \circ a_1) + \phi\left(\frac{1}{2} t_{\frac{1}{2}} \circ a_{\frac{1}{2}}\right).$$

For p, applying a standard argument to the equation above, we get

$$\begin{aligned}\phi\left(\frac{1}{2} t_{\frac{1}{2}} \circ s_1 + \frac{1}{2} p \circ (t_{\frac{1}{2}} \circ s_{\frac{1}{2}})\right) &= \phi\left(p \circ \left(t_{\frac{1}{2}} \circ s_1 + \frac{1}{2} t_{\frac{1}{2}} \circ s_{\frac{1}{2}}\right)\right) \\ &= \phi(p \circ (t_{\frac{1}{2}} \circ a_1)) + \phi\left(\frac{1}{2} p \circ (t_{\frac{1}{2}} \circ a_{\frac{1}{2}})\right) \\ &= \phi(\frac{1}{2} t_{\frac{1}{2}} \circ a_1) + \phi\left(\frac{1}{2} p \circ (t_{\frac{1}{2}} \circ a_{\frac{1}{2}})\right).\end{aligned} \tag{3.3}$$

For $t_0 \in A_0$, applying a standard argument to (3.3), we have that

$$\begin{aligned}&\phi\left(\frac{1}{2} t_0 \circ (t_{\frac{1}{2}} \circ s_1) + \frac{1}{2} t_0 \circ (p \circ (t_{\frac{1}{2}} \circ s_{\frac{1}{2}}))\right) \\ &\quad = \phi\left(\frac{1}{2} t_0 \circ (t_{\frac{1}{2}} \circ a_1)\right) + \phi\left(\frac{1}{2} t_0 \circ (p \circ (t_{\frac{1}{2}} \circ a_{\frac{1}{2}}))\right).\end{aligned} \tag{3.4}$$

By Lemmas 2.3 and 2.4, we see that $t_0 \circ (p \circ (t_{\frac{1}{2}} \circ s_{\frac{1}{2}})) = t_0 \circ (p \circ (t_{\frac{1}{2}} \circ a_{\frac{1}{2}})) = 0$. Thus (3.4) implies

$$\phi\left(\frac{1}{2}t_0 \circ (t_{\frac{1}{2}} \circ s_1)\right) = \phi\left(\frac{1}{2}t_0 \circ (t_{\frac{1}{2}} \circ a_1)\right).$$

Therefore, $\frac{1}{2}t_0 \circ (t_{\frac{1}{2}} \circ s_1) = \frac{1}{2}t_0 \circ (t_{\frac{1}{2}} \circ a_1)$ for all $t_{\frac{1}{2}} \in A_{\frac{1}{2}}$ and $t_0 \in A_0$. It follows from Theorem 3.1(i, iii) that $s_1 = a_1$.

For t_0, applying a standard argument to (3.1), we get

$$\begin{aligned}\phi(s_0 \circ t_0 + s_{\frac{1}{2}} \circ t_0) &= \phi(s \circ t_0) = \phi(a_1 \circ t_0) + \phi(a_{\frac{1}{2}} \circ t_0) + \phi(a_0 \circ t_0)\\ &= \phi(a_{\frac{1}{2}} \circ t_0) + \phi(a_0 \circ t_0) + \phi(0) = \phi(a_{\frac{1}{2}} \circ t_0) + \phi(a_0 \circ t_0). \end{aligned} \tag{3.5}$$

For $t_{\frac{1}{2}}$, applying a standard argument to (3.5), we have

$$\phi((s_0 \circ t_0) \circ t_{\frac{1}{2}} + (s_{\frac{1}{2}} \circ t_0) \circ t_{\frac{1}{2}}) = \phi((a_{\frac{1}{2}} \circ t_0) \circ t_{\frac{1}{2}}) + \phi((a_0 \circ t_0) \circ t_{\frac{1}{2}}). \tag{3.6}$$

Since $(s_0 \circ t_0) \circ t_{\frac{1}{2}}, (a_0 \circ t_0) \circ t_{\frac{1}{2}} \in A_{\frac{1}{2}}$ and $(s_{\frac{1}{2}} \circ t_0) \circ t_{\frac{1}{2}}, (a_{\frac{1}{2}} \circ t_0) \circ t_{\frac{1}{2}} \in A_1 \oplus A_0$, we have $p \circ ((s_0 \circ t_0) \circ t_{\frac{1}{2}}) = \frac{1}{2}((s_0 \circ t_0) \circ t_{\frac{1}{2}})$, $p \circ ((a_0 \circ t_0) \circ t_{\frac{1}{2}}) = \frac{1}{2}((a_0 \circ t_0) \circ t_{\frac{1}{2}})$, and $p \circ ((s_{\frac{1}{2}} \circ t_0) \circ t_{\frac{1}{2}})$, $p \circ ((a_{\frac{1}{2}} \circ t_0) \circ t_{\frac{1}{2}}) \in A_1$. For p, applying a standard argument to (3.6), we get

$$\begin{aligned}\phi&\left(\frac{1}{2}((s_0 \circ t_0) \circ t_{\frac{1}{2}}) + p \circ ((s_{\frac{1}{2}} \circ t_0) \circ t_{\frac{1}{2}})\right) = \phi(p \circ ((s_0 \circ t_0) \circ t_{\frac{1}{2}} + (s_{\frac{1}{2}} \circ t_0) \circ t_{\frac{1}{2}}))\\ &= \phi(p \circ ((a_0 \circ t_0) \circ t_{\frac{1}{2}})) + \phi(p \circ ((a_{\frac{1}{2}} \circ t_0) \circ t_{\frac{1}{2}}))\\ &= \phi\left(\frac{1}{2}((a_0 \circ t_0) \circ t_{\frac{1}{2}})\right) + \phi(p \circ ((a_{\frac{1}{2}} \circ t_0) \circ t_{\frac{1}{2}})). \end{aligned} \tag{3.7}$$

For $t_0' \in A_0$, since $p \circ ((s_{\frac{1}{2}} \circ t_0) \circ t_{\frac{1}{2}}), p \circ ((a_{\frac{1}{2}} \circ t_0) \circ t_{\frac{1}{2}}) \in A_1$, we have that $t_0' \circ (p \circ ((s_{\frac{1}{2}} \circ t_0) \circ t_{\frac{1}{2}})) = 0$ and $t_0' \circ (p \circ ((a_{\frac{1}{2}} \circ t_0) \circ t_{\frac{1}{2}})) = 0$. Thus for $t_0' \in A_0$, applying a standard argument to (3.7), we have that

$$\begin{aligned}\phi&\left(\frac{1}{2}((s_0 \circ t_0) \circ t_{\frac{1}{2}}) \circ t_0'\right)\\ &= \phi\left(\left(\frac{1}{2}((s_0 \circ t_0) \circ t_{\frac{1}{2}}) + p \circ ((s_{\frac{1}{2}} \circ t_0) \circ t_{\frac{1}{2}})\right)\right) \circ t_0'\\ &= \phi\left(\frac{1}{2}((a_0 \circ t_0) \circ t_{\frac{1}{2}}) \circ t_0'\right) + \phi((p \circ ((a_{\frac{1}{2}} \circ t_0) \circ t_{\frac{1}{2}})) \circ t_0')\\ &= \phi\left(\frac{1}{2}((a_0 \circ t_0) \circ t_{\frac{1}{2}}) \circ t_0'\right). \end{aligned}$$

Therefore, $\frac{1}{2}((s_0 \circ t_0) \circ t_{\frac{1}{2}}) \circ t_0' = \frac{1}{2}((a_0 \circ t_0) \circ t_{\frac{1}{2}}) \circ t_0'$ for all $t_{\frac{1}{2}} \in A_{\frac{1}{2}}$ and $t_0, t_0' \in A_0$. It follows from Theorem 3.1(i, ii, iii) that $s_0 = a_0$. Thus $s = a_1 + a_{\frac{1}{2}} + a_0$. $\square$

Lemma 3.4. *Let $a_{\frac{1}{2}}, b_{\frac{1}{2}} \in A_{\frac{1}{2}}$ and $a_0 \in A_0$. Then $\phi(a_{\frac{1}{2}} \circ a_0 + b_{\frac{1}{2}}) = \phi(a_{\frac{1}{2}} \circ a_0) + \phi(b_{\frac{1}{2}})$.*

Proof. Compute

$$(2p + a_{\frac{1}{2}}) \circ (a_0 + b_{\frac{1}{2}}) = 2p \circ b_{\frac{1}{2}} + 2p \circ a_0 + b_{\frac{1}{2}} \circ a_{\frac{1}{2}} + a_{\frac{1}{2}} \circ a_0 = b_{\frac{1}{2}} + b_{\frac{1}{2}} \circ a_{\frac{1}{2}} + a_{\frac{1}{2}} \circ a_0.$$

By Lemmas 3.3 and 3.2, we have $\phi(c_0 + c_1) = \phi(c_0) + \phi(c_1)$ for all $c_0 \in A_0$ and $c_1 \in A_1$. Hence it follows from Lemma 3.3 that $\phi(c_0 + c_1 + c_{\frac{1}{2}}) = \phi(c_0) + \phi(c_1) + \phi(c_{\frac{1}{2}}) = \phi(c_0 + c_1) + \phi(c_{\frac{1}{2}})$ for all $c_0 \in A_0$, $c_1 \in A_1$ and $c_{\frac{1}{2}} \in A_{\frac{1}{2}}$. By Lemma 2.4, we see that $a_{\frac{1}{2}} \circ a_0 \in A_{\frac{1}{2}}$ and $b_{\frac{1}{2}} \circ a_{\frac{1}{2}} \in A_0 \oplus A_1$. Hence $b_{\frac{1}{2}} + a_{\frac{1}{2}} \circ a_0 \in A_{\frac{1}{2}}$. Consequently

$$\begin{aligned}\phi(b_{\frac{1}{2}} + a_{\frac{1}{2}} \circ a_0) + \phi(b_{\frac{1}{2}} \circ a_{\frac{1}{2}}) &= \phi(b_{\frac{1}{2}} + a_{\frac{1}{2}} \circ a_0 + b_{\frac{1}{2}} \circ a_{\frac{1}{2}}) \\ &= \phi((2p + a_{\frac{1}{2}}) \circ (a_0 + b_{\frac{1}{2}})) \\ &= (\phi(2p) + \phi(a_{\frac{1}{2}})) \circ (\phi(a_0) + \phi(b_{\frac{1}{2}})) \\ &= \phi(2p) \circ \phi(a_0) + \phi(a_{\frac{1}{2}}) \circ \phi(a_0) + \phi(2p) \circ \phi(b_{\frac{1}{2}}) + \phi(a_{\frac{1}{2}}) \circ \phi(b_{\frac{1}{2}}) \\ &= \phi(2p \circ a_0) + \phi(a_{\frac{1}{2}} \circ a_0) + \phi(2p \circ b_{\frac{1}{2}}) + \phi(a_{\frac{1}{2}} \circ b_{\frac{1}{2}}) \\ &= \phi(a_{\frac{1}{2}} \circ a_0) + \phi(b_{\frac{1}{2}}) + \phi(a_{\frac{1}{2}} \circ b_{\frac{1}{2}}).\end{aligned}$$

Thus, $\phi(a_{\frac{1}{2}} \circ a_0 + b_{\frac{1}{2}}) = \phi(a_{\frac{1}{2}} \circ a_0) + \phi(b_{\frac{1}{2}})$.

Lemma 3.5. *Let $a_{\frac{1}{2}}, b_{\frac{1}{2}} \in A_{\frac{1}{2}}$. Then $\phi(a_{\frac{1}{2}} + b_{\frac{1}{2}}) = \phi(a_{\frac{1}{2}}) + \phi(b_{\frac{1}{2}})$.*

Proof. Choose $s = s_1 + s_{\frac{1}{2}} + s_0 \in A$ such that

$$\phi(s) = \phi(a_{\frac{1}{2}}) + \phi(b_{\frac{1}{2}}). \tag{3.8}$$

For $t_0 \in A_0$, applying a standard argument to (3.8) and using Lemma 3.4, we get

$$\phi(s \circ t_0) = \phi(a_{\frac{1}{2}} \circ t_0) + \phi(b_{\frac{1}{2}} \circ t_0) = \phi(a_{\frac{1}{2}} \circ t_0 + b_{\frac{1}{2}} \circ t_0).$$

Hence $(s_0 + s_{\frac{1}{2}}) \circ t_0 = s \circ t_0 = (a_{\frac{1}{2}} + b_{\frac{1}{2}}) \circ t_0$ for every $t_0 \in A_0$. Since $s_0 \circ t_0 \in A_0$ and $s_{\frac{1}{2}} \circ t_0, (a_{\frac{1}{2}} + b_{\frac{1}{2}}) \circ t_0 \in A_{\frac{1}{2}}$, we have $s_0 \circ t_0 = 0$ and $s_{\frac{1}{2}} \circ t_0 = (a_{\frac{1}{2}} + b_{\frac{1}{2}}) \circ t_0$ for every $t_0 \in A_0$. By Theorem 3.1(ii) and (iii), we obtain that $s_0 = 0$ and $s_{\frac{1}{2}} = a_{\frac{1}{2}} + b_{\frac{1}{2}}$.

For $t_{\frac{1}{2}}$, applying a standard argument to (3.8) again, we have

$$\phi(s_1 \circ t_{\frac{1}{2}} + s_{\frac{1}{2}} \circ t_{\frac{1}{2}}) = \phi(s \circ t_{\frac{1}{2}}) = \phi(a_{\frac{1}{2}} \circ t_{\frac{1}{2}}) + \phi(b_{\frac{1}{2}} \circ t_{\frac{1}{2}}).$$

For p, applying a standard argument to the equation above, we get that

$$\begin{aligned}\phi\left(\frac{1}{2}s_1 \circ t_{\frac{1}{2}} + p \circ (s_{\frac{1}{2}} \circ t_{\frac{1}{2}})\right) &= \phi(p \circ (s_1 \circ t_{\frac{1}{2}} + s_{\frac{1}{2}} \circ t_{\frac{1}{2}})) \\ &= \phi(p \circ (a_{\frac{1}{2}} \circ t_{\frac{1}{2}})) + \phi(p \circ (b_{\frac{1}{2}} \circ t_{\frac{1}{2}})).\end{aligned} \tag{3.9}$$

For $t_0 \in A_0$, applying a standard argument to (3.9), we have

$$\begin{aligned}\phi\left(\frac{1}{2}(s_1 \circ t_{\frac{1}{2}}) \circ t_0\right) &= \phi\left(\left(\frac{1}{2}s_1 \circ t_{\frac{1}{2}} + (p \circ (s_{\frac{1}{2}} \circ t_{\frac{1}{2}}))\right) \circ t_0\right) \\ &= \phi((p \circ (a_{\frac{1}{2}} \circ t_{\frac{1}{2}})) \circ t_0) + \phi((p \circ (b_{\frac{1}{2}} \circ t_{\frac{1}{2}})) \circ t_0) = 2\phi(0) = 0.\end{aligned}$$

Hence $(s_1 \circ t_{\frac{1}{2}}) \circ t_0 = 0$ for every $t_{\frac{1}{2}} \in A_{\frac{1}{2}}$ and $t_0 \in A_0$. By Theorem 3.1(iii, i), it follows that $s_1 = 0$. Consequently, $s = s_{\frac{1}{2}} = a_{\frac{1}{2}} + b_{\frac{1}{2}}$. $\square$

Lemma 3.6. *Let $a_1, b_1 \in A_1$. Then $\phi(a_1 + b_1) = \phi(a_1) + \phi(b_1)$.*

Proof. Choose $s = s_1 + s_{\frac{1}{2}} + s_0 \in A$ such that

$$\phi(s) = \phi(a_1) + \phi(b_1). \tag{3.10}$$

For $t_0 \in A_0$, applying a standard argument to (3.10), we see that

$$\phi(s_{\frac{1}{2}} \circ t_0 + s_0 \circ t_0) = \phi(s \circ t_0) = \phi(a_1 \circ t_0) + \phi(b_1 \circ t_0) = 2\phi(0) = 0.$$

Therefore, $s_{\frac{1}{2}} \circ t_0 + s_0 \circ t_0 = 0$ for every $t_0 \in A_0$. Since $s_{\frac{1}{2}} \circ t_0 \in A_{\frac{1}{2}}$ and $s_0 \circ t_0 \in A_0$, we get that $s_{\frac{1}{2}} \circ t_0 = 0$ and $s_0 \circ t_0 = 0$ for every $t_0 \in A_0$. By Theorem 3.1(ii, iii), we have that $s_{\frac{1}{2}} = 0$ and $s_0 = 0$. So $s = s_1$.

Now there remains to prove that $s_1 = a_1 + b_1$. For $t_{\frac{1}{2}}$, applying a standard argument to (3.10) again and using Lemma 3.5, we get

$$\phi(s_1 \circ t_{\frac{1}{2}}) = \phi(s \circ t_{\frac{1}{2}}) = \phi(a_1 \circ t_{\frac{1}{2}}) + \phi(b_1 \circ t_{\frac{1}{2}}) = \phi(a_1 \circ t_{\frac{1}{2}} + b_1 \circ t_{\frac{1}{2}}).$$

Therefore, $s_1 \circ t_{\frac{1}{2}} = (a_1 + b_1) \circ t_{\frac{1}{2}}$ for every $t_{\frac{1}{2}} \in A_{\frac{1}{2}}$. It follows from Theorem 3.1(i) that $s_1 = a_1 + b_1$. Consequently $s = a_1 + b_1$. $\square$

Lemma 3.7. *Let $a_0, b_0 \in A_0$. Then $\phi(a_0 + b_0) = \phi(a_0) + \phi(b_0)$.*

Proof. Choose $s = s_1 + s_{\frac{1}{2}} + s_0 \in A$ such that

$$\phi(s) = \phi(a_0) + \phi(b_0). \tag{3.11}$$

For p, applying a standard argument to (3.11), we see that

$$\phi\left(\frac{1}{2}s_{\frac{1}{2}} + s_1\right) = \phi(s \circ p) = \phi(a_0 \circ p) + \phi(b_0 \circ p) = 2\phi(0) = 0.$$

Therefore, $\frac{1}{2}s_{\frac{1}{2}} + s_1 = 0$. Hence $s_{\frac{1}{2}} = 0$, $s_1 = 0$ and $s = s_0$.

Now there remains to prove that $s_0 = a_0 + b_0$. For $t_{\frac{1}{2}}$, applying a standard argument to (3.11) again, we get by Lemma 3.5 that

$$\phi(s_0 \circ t_{\frac{1}{2}}) = \phi(a_0 \circ t_{\frac{1}{2}}) + \phi(b_0 \circ t_{\frac{1}{2}}) = \phi(a_0 \circ t_{\frac{1}{2}} + b_0 \circ t_{\frac{1}{2}}).$$

Therefore, $s_0 \circ t_{\frac{1}{2}} = (a_0 + b_0) \circ t_{\frac{1}{2}}$ for every $t_{\frac{1}{2}}$. It follows from Theorem 3.1(i) that $s_0 = a_0 + b_0$. Consequently $s = a_0 + b_0$. $\square$

Proof of Theorem 3.1. Let $a = a_1 + a_{\frac{1}{2}} + a_0$, $b = b_1 + b_{\frac{1}{2}} + b_0 \in A$. Then Lemmas 3.3–3.7 are all used in seeing the equalities

$$\begin{aligned}
\phi(a+b) &= \phi((a_1 + b_1) + (a_{\frac{1}{2}} + b_{\frac{1}{2}}) + (a_0 + b_0)) \\
&= \phi(a_1 + b_1) + \phi(a_{\frac{1}{2}} + b_{\frac{1}{2}}) + \phi(a_0 + b_0) \\
&= \phi(a_1) + \phi(b_1) + \phi(a_{\frac{1}{2}}) + \phi(b_{\frac{1}{2}}) + \phi(a_0) + \phi(b_0) \\
&= \phi(a_1 + a_{\frac{1}{2}} + a_0) + \phi(b_1 + b_{\frac{1}{2}} + b_0) \\
&= \phi(a) + \phi(b)
\end{aligned}$$

hold true. That is, ϕ is additive on A. $\square$

4. Corollaries

The following corollary generalizes the results in [9]. Here it does not need the condition (iii) in [9, Theorem 1.1].

Corollary 4.1. *Let A and A' be algebras over the field Q of rational numbers. Suppose that A contains an idempotent e_1 which satisfies*

(i) *$e_ixe_jAe_l = 0$ or $e_lAe_ixe_j = 0$ implies $e_ixe_j = 0$ $(1 \leqslant i, j, l \leqslant 2)$, where $e_2 = 1 - e_1$ (A need not have an identity element).*

(ii) *If* $e_2te_2xe_2 + e_2xe_2te_2 = 0$ *for each* $t \in A$, *then* $e_2xe_2 = 0$.

Let k be a fixed nonzero rational number. Suppose that $\phi : A \longrightarrow A'$ *is a k-Jordan map, i.e.* ϕ *is bijective map satisfying* $\phi(k(ab + ba) = k(\phi(a)\phi(b) + \phi(b)\phi(a))$ *for all* $a, b \in A$. *Then* ϕ *is additive.*

Proof. For $a, b \in A$ or A', let $a \circ b = k(ab + ba)$. Then $(A, \circ)$ and $(A', \circ)$ are Jordan algebras, and $\phi : (A, \circ) \longrightarrow (A', \circ)$ is a Jordan map from $(A, \circ)$ onto $(A', \circ)$. Clearly, if the idempotent e_1 satisfies the conditions (i) and (ii), then A and e_1 satisfy the conditions of Theorem 3.1. Hence ϕ is additive. $\square$

Corollary 4.2 [7]. *Let F be the real number or the complex number field. Let A be a standard subalgebra of a nest algebra* $\mathcal{T}(\mathcal{N})$ *on a Hilbert space H over F of dimension greater than* 1 *and R be an algebra over the field of rational numbers. Let k be a nonzero rational number. Suppose* $\phi : A \longrightarrow R$ *is a bijective map* $\phi(k(ab + ba) = k(\phi(a)\phi(b) + \phi(b)\phi(a))$ *for all* $a, b \in A$. *Then* ϕ *is additive.*

Proof. As the proof of Corollary 4.1, for $a, b \in A$ or B, let $a \circ b = k(ab + ba)$. Then $(A, \circ)$ and $(B, \circ)$ are Jordan algebras, and $\phi : (A, \circ) \longrightarrow (B, \circ)$ is a Jordan map from $(A, \circ)$ onto $(B, \circ)$. By Lemmas 2–4 of [7], there is an idempotent in A satisfying the conditions of Theorem 3.1. Hence ϕ is additive. $\square$

Corollary 4.3. *Let A be a unital Jordan algebra and p a non-trivial idempotent in A. Let B be a Jordan algebras. Let* $A = A_1 \oplus A_{\frac{1}{2}} \oplus A_0$ *be the Peirce decomposition of A with respect to p. If A satisfies the following conditions:*

(i) *Let* $a_i \in A_i$ $(i = 1, 0))$. *If* $a_i \circ t_{\frac{1}{2}} = 0$ *for all* $t_{\frac{1}{2}} \in A_{\frac{1}{2}}$, *then* $a_i = 0$,

then every Jordan map from A onto B is additive.

Proof. Let $a_0 \in A_0$. If $a_0 \circ t_0 = 0$ for all $t_0 \in A_0$, since $(1 - p) \in A_0$, we have $a_0 = a_0 \circ (1 - p) = 0$. Thus the condition (ii) of Theorem 3.1 holds.

Let $a_{\frac{1}{2}} \in A_{\frac{1}{2}}$. If $a_{\frac{1}{2}} \circ t_0 = 0$ for all $t_0 \in A_0$, since $2(1 - p) \in A_0$, we get that $a_{\frac{1}{2}} = 2(1 - p) \circ a_{\frac{1}{2}} = 0$. Thus the condition (iii) of Theorem 3.1 holds. By Theorem 3.1, every Jordan map from A onto B is additive. $\square$

For a Hilbert space H, we write $B(H)$ and $F(H)$ for the algebra of all linear bounded operators on H and the algebra of all finite rank operators on H, respectively. For $a \in B(H)$, denote by a^* the adjoint operator of a. If $a = a^*$, a is called a self-adjoint operator on H. Denote by $B(H)_s$ and $F(H)_s$ the real linear space of all self-adjoint operators on H and the real linear space of all self-adjoint finite rank operators on H, respectively. Clearly, $B(H)_s$ and $F(H)_s$ are speacial Jordan subalgebra of $B(H)$. A Jordan subalgebra of $B(H)_s$ is called a standard Jordan operator algebra on H if it contains $F(H)_s$.

Theorem 4.4 [6]. *Let H be a Hilbert space of dimension* > 1 *and suppose that A is a standard Jordan operator algerba on H. Let B be an arbitrary Jordan algebra. Then every Jordan map* ϕ *from A onto B is additive.*

Proof. Lemma 2.3 of [6], there is an idempotent in A satisfying the conditions of Theorem 3.1. Hence ϕ is additive. $\square$

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